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Domain Of Convergence Of Power Series And The Application Of Taylor Series Expansions To Approximate Computations

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Abstract

Power series and Taylor expansions remain among the most widely used tools for the approximate evaluation of functions, constants and definite integrals, yet in practice their use is frequently detached from the question that ultimately governs their efficiency, namely the position of the evaluation point relative to the domain of convergence. This paper examines the link between the radius and interval of convergence of a power series and the accuracy and cost of Taylor-based approximate computations. The Cauchy–Hadamard formula and the d'Alembert ratio criterion are used to determine the radius of convergence of six representative series, and finite-index estimates of the radius are compared with exact values. Taylor polynomials with Lagrange and alternating-series remainder bounds are then applied to the approximate evaluation of e^0.5, sin0.3, ln1.2, √1.1 and of two non-elementary integrals, ∫_0^1▒e^(-x^2) dx and ∫_0^1▒sinx/x dx. All computations were carried out in IEEE 754 double precision and verified against reference values. The results show that the a priori remainder bounds overestimate the true error by a factor of only 1.0–1.6 for points well inside the interval of convergence, and that the number of terms required for a fixed tolerance grows approximately as ln(1/ε)/ln(R/|x-x_0 |), becoming unbounded as the evaluation point approaches the boundary. Rewriting the target quantity so that the series is evaluated at a point with a small ratio |x-x_0 |/R reduces the work by up to eleven orders of magnitude: ln2 to 10^(-12) requires about 5×10^11 terms of the Mercator series but only 12 terms of the inverse hyperbolic tangent series, and π requires about 10^12 terms of the Leibniz series but only 9 terms of Machin's formula. A five-step procedure that places the analysis of the domain of convergence before the choice of the number of terms is proposed for teaching and for routine engineering computation..

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Introduction

The representation of a function by a power series is one of the central constructions of mathematical analysis, because it reduces the study of a transcendental function to operations on polynomials, which can be evaluated using only addition and multiplication. A power series centred at a point x_0 has the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n$, and its fundamental property is that the set of points at which it converges is always an interval symmetric about x_0, possibly degenerating to a single point or extending to the whole real line [1]. Once the coefficients are taken to be $a_n = f^{(n)}(x_0)/n!$, the series becomes the Taylor series of f, and its partial sums, the Taylor polynomials, provide the most natural local approximation of a smooth function.

In the university curriculum, the topics of convergence of power series and of Taylor expansion are usually taught in sequence but treated as largely separate subjects: the first is presented as an exercise in applying the ratio or root test, the second as a technique for writing down expansions of elementary functions and estimating the remainder [2]. Students therefore learn to find an interval of convergence and, separately, to approximate $\ln 1.2$ or $\sqrt[3]{30}$, but they rarely see how the first result determines the feasibility and cost of the second. In engineering practice, the same disconnect leads to computational schemes that converge in principle but are useless in practice, such as the evaluation of $\ln 2$ directly from the series for $\ln(1 + x)$ at $x = 1$.

The classical theory provides a complete answer to the question of where a power series converges. The Cauchy–Hadamard theorem expresses the radius of convergence R through the upper limit of $|a_n|^{1/n}$, while Abel's lemma guarantees absolute convergence for $|x - x_0| < R$ and divergence for $|x - x_0| > R$; behaviour at the two endpoints must be examined separately and can differ between them [3]. Inside the interval, the series converges uniformly on every closed subinterval, so it can be differentiated and integrated term by term without changing the radius, a property that underlies the expansion of non-elementary integrals into series [4].

A deeper explanation of the size of R comes from complex analysis. The Taylor series of an analytic function about x_0 converges in the largest open disc centred at x_0 that contains no singularity of the function, so the radius of convergence equals the distance from x_0 to the nearest singular point in the complex plane [5]. This explains why the series for $1/(1 + x^2)$, a function that is infinitely differentiable on the entire real axis, nevertheless has $R = 1$: the obstruction lies at the complex points $\pm i$. The same principle shows that convergence is a property of the expansion point as much as of the function, which is precisely what makes it a design parameter in approximate computation.

Outside the interval of convergence, the partial sums of a power series are not merely inaccurate; they diverge, and no increase in the number of terms can improve them. Although summation methods for divergent series exist and have a long history, they do not alter the ordinary partial sums used in elementary approximation and lie outside routine computational practice [6]. For the practitioner, therefore, the domain of convergence is a hard boundary that must be identified before any approximation is attempted.

The aim of this study is to quantify the relationship between the domain of convergence of a power series and the efficiency of Taylor-based approximate computation. The specific objectives are: (i) to compare exact radii of convergence with finite-index estimates produced by the ratio and root criteria; (ii) to evaluate the accuracy of Taylor approximations of elementary functions and non-elementary integrals and compare the true errors with a priori remainder bounds; (iii) to measure how the number of terms required for a given tolerance depends on the relative position of the evaluation point within the interval of convergence; and (iv) to demonstrate how reformulating a problem so as to move the evaluation point toward the centre of the interval reduces computational cost.

Materials and Methods

Determination of the domain of convergence. For a power series $\sum a_n(x-x_0)^n$ the radius of convergence was defined by the Cauchy–Hadamard formula

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \qquad (1)$$

with the conventions $R = \infty$ when the upper limit is zero and $R = 0$ when it is infinite. When the limit of the ratio of consecutive coefficients exists, the d'Alembert form

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad (2)$$

was used as the primary analytical tool, and the root form (1) as an independent check. The series converges absolutely for $|x-x_0| < R$ and diverges for $|x-x_0| > R$. At the endpoints $x = x_0 \pm R$ the resulting numerical series were investigated separately using the comparison test with p-series, the Leibniz criterion for alternating series and the necessary condition $a_n(x-x_0)^n \to 0$. To show how quickly the limits (1) and (2) are approached, the finite-index quantities $\rho_n = |a_n/a_{n+1}|$ and $r_n = |a_n|^{-1/n}$ were computed for $n = 10$, 50 and 100 for six test series chosen to represent all qualitatively different cases: finite R with convergence at one endpoint, finite R with convergence at both endpoints, $R = \infty$, a shifted centre, divergence at both endpoints, and a non-unit radius with convergence at one endpoint.

Taylor expansion and remainder estimates. If f has $n+1$ continuous derivatives on an interval containing x_0 and x, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x), \qquad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \qquad (3)$$

for some ξ between x_0 and x. The Taylor series represents f at the point x if and only if $R_n(x) \to 0$; convergence of the series alone is not sufficient, as the classical example $f(x) = e^{-1/x^2}$, $f(0) = 0$, shows, since all its Maclaurin coefficients vanish. Three types of a priori error bound were used, depending on the structure of the series. For series with explicitly bounded derivatives (e^x, $\sin x$) the Lagrange form (3) was bounded by $M_{n+1}|x-x_0|^{n+1}/(n+1)!$, where $M_{n+1} = \max\left|f^{(n+1)}\right|$ on the relevant interval. For alternating series whose terms decrease monotonically in absolute value, the bound

$$|S - S_n| \le |u_{n+1}| \qquad (4)$$

was applied, where u_{n+1} is the first omitted term. For series with positive terms whose ratio satisfies $|u_{k+1}/u_k| \le q < 1$ for all $k \ge n+1$, the geometric tail bound $|S-S_n| \le |u_{n+1}|/(1-q)$ was used. These estimates follow the standard treatment of truncation error in numerical analysis [7].

Test problems. Four elementary function values were approximated by Maclaurin polynomials: $e^{0.5}$ from $e^x = \sum x^k/k!$; $\sin 0.3$ from $\sin x = \sum(-1)^k x^{2k+1}/(2k+1)!$; $\ln 1.2$ from the Mercator series $\ln(1+x) = \sum_{k\ge1}(-1)^{k+1}x^k/k$ at $x = 0.2$; and $\sqrt{1.1}$ from the binomial series $(1+x)^{1/2} = \sum\binom{1/2}{k}x^k$ at $x = 0.1$. Two non-elementary integrals were evaluated by term-by-term integration of the Maclaurin series of the integrand, which is permitted because the series converge uniformly on [0,1]:

$$\int_0^1 e^{-x^2}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(2k+1)}, \qquad \int_0^1 \frac{\sin x}{x}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\,(2k+1)!}. \qquad (5)$$

Convergence-rate and reformulation experiments. To isolate the effect of the position of the evaluation point, the absolute error $|\ln(1+x) - T_N(x)|$ of the Mercator series was computed for $N = 1, \ldots, 60$ at $x = 0.2$, 0.5, 0.8 and 1.0, corresponding to relative distances $|x-x_0|/R$ of 0.2, 0.5, 0.8 and 1.0. Three classical constants were then computed in two ways each. For $\ln 2$, the Mercator series at the endpoint $x = 1$ was compared with the identity

$$\ln 2 = 2\,\operatorname{artanh}\,1/3 = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)\,3^{2k+1}}, \qquad (6)$$

which follows from $\ln\frac{1+t}{1-t} = 2\,\operatorname{artanh} t$ with $t = 1/3$. For π, the Leibniz series $\pi = 4\arctan 1$ was compared with Machin's formula $\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}$. For e, the Maclaurin series of e^x at $x = 1$ served as a reference case with $R = \infty$. For each method the smallest number of terms giving an absolute error below 10^{-6} and below 10^{-12} was determined; for the slowly convergent series it was obtained from the asymptotic error $1/(2N)$ for the Mercator series at $x = 1$ and $1/N$ for the Leibniz series, and confirmed by direct summation of 10^6 and 2×10^6 terms respectively.

Computational environment. All computations were performed in Python 3 with the NumPy and SciPy libraries in IEEE 754 binary64 arithmetic, which carries a unit roundoff of $2^{-53} \approx 1.1 \times 10^{-16}$ [8]. Reference values were taken from the library functions exp, sin, log1p, sqrt, erf and sici, the last two giving $\int_0^1 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(1)$ and the sine integral $\mathrm{Si}(1)$. Partial sums were accumulated in order of decreasing term magnitude, which limits the growth of rounding error in long recursive summations [9]. Figures were produced with Matplotlib at 300 dpi.

Results

Table 1 summarises the domain of convergence of the six test series together with the finite-index estimates of the radius. For every series with finite nonzero R, both ρ_n and r_n approach the exact value, but at noticeably different rates. The ratio estimate converges with an error of order $1/n$: for $\sum x^n/n$ it equals 1.10, 1.02 and 1.01 at $n = 10$, 50 and 100. The root estimate converges considerably more slowly, with an error of order $(\ln n)/n$, and at $n = 100$ still exceeds the true radius by 4.7 % for $\sum x^n/n$ and by 9.7 % for $\sum x^n/n^2$. For $\sum x^n/n!$ both quantities grow without bound, the ratio estimate exactly as $n + 1$, confirming $R = \infty$. The series $\sum n^2 x^n/2^n$ approaches $R = 2$ from below, and the central binomial series approaches $R = 1/4$ from above, so the direction of approach is not universal.

Table 1. Radius and interval of convergence of the test series and finite-index estimates

Series	Exact R	ρ_{10}	ρ_{50}	ρ_{100}	r_{100}	Interval of convergence
$\sum x^n/n$	1	1.1000	1.0200	1.0100	1.0471	[−1, 1)
$\sum x^n/n^2$	1	1.2100	1.0404	1.0201	1.0965	[−1, 1]
$\sum x^n/n!$	∞	11.000	51.000	101.00	37.993	(−∞, ∞)
$\sum (x-2)^n/(3^n n)$	3	3.3000	3.0600	3.0300	3.1414	[−1, 5)
$\sum n^2 x^n/2^n$	2	1.6529	1.9223	1.9606	1.8240	(−2, 2)
$\sum \binom{2n}{n} x^n$	1/4	0.2619	0.2525	0.2512	0.2573	[−1/4, 1/4)

The endpoint analysis confirmed that endpoint behaviour cannot be inferred from R alone. Three series sharing a radius of the same kind behave differently at the boundary: $\sum x^n/n$ converges conditionally at $x = -1$ and diverges at $x = 1$; $\sum x^n/n^2$ converges absolutely at both endpoints by comparison with $\sum 1/n^2$; and $\sum n^2 x^n/2^n$ diverges at both endpoints because its general term does not tend to zero. For the central binomial series, $\binom{2n}{n}4^{-n} \sim 1/\sqrt{\pi n}$, so the series diverges at $x = 1/4$ but converges by the Leibniz criterion at $x = -1/4$.

Table 2 reports the Taylor approximations of the four function values and two integrals. In every case the true absolute error lies below the a priori bound, as theory requires, and the ratio of bound to true error remains small. For $\sin 0.3$ and the sine integral the ratio does not exceed 1.02, because consecutive terms decrease so rapidly that the first omitted term almost equals the entire tail. For $e^{0.5}$ the ratio is

approximately 1.5 to 1.6, reflecting the replacement of e^{ξ}, $\xi \in (0,0.5)$, by its maximum $e^{0.5}$ in the Lagrange form. Approximations at points close to the centre are highly economical: a seventh-degree polynomial, containing only four nonzero terms, gives sin0.3 with an error of 5.4×10^{-11}, and five terms of the binomial series give $\sqrt{1.1}$ to 2.5×10^{-7}. The two non-elementary integrals, which have no closed form in elementary functions, are obtained to eight or nine correct digits with ten and five terms respectively.

Table 2. Taylor approximations: number of terms, approximate value, true error and a priori error bound

Quantity	Expansion point / $\lvert x - x_0 \rvert / R$	n	Approximation	True error	A priori bound
$e^{0.5}$	$x = 0.5$ / 0	4	1.6484375000	2.84×10^{-4}	4.29×10^{-4}
		8	1.6487212650	5.66×10^{-9}	8.87×10^{-9}
sin0.3	$x = 0.3$ / 0	3	0.2955000000	2.02×10^{-5}	2.03×10^{-5}
		7	0.2955202066	5.42×10^{-11}	5.42×10^{-11}
ln1.2	$x = 0.2$ / 0.2	4	0.1822666667	5.49×10^{-5}	6.40×10^{-5}
		8	0.1823215086	4.82×10^{-8}	5.69×10^{-8}
$\sqrt{1.1}$	$x = 0.1$ / 0.1	2	1.0487500000	5.89×10^{-5}	6.25×10^{-5}
		4	1.0488085938	2.54×10^{-7}	2.73×10^{-7}
$\int_0^1 e^{-x^2}\,dx$	— / 0	5	0.7467291967	9.49×10^{-5}	1.07×10^{-4}
		9	0.7468241207	1.21×10^{-8}	1.31×10^{-8}
$\int_0^1 \frac{\sin x}{x}\,dx$	— / 0	2	0.9461111111	2.80×10^{-5}	2.83×10^{-5}
		4	0.9460830726	2.27×10^{-9}	2.28×10^{-9}

Note: for e^x, $\ln(1+x)$ and $\sqrt{1+x}$, n is the degree of the Taylor polynomial; for $\sin x$, n is the degree $2m+1$; for the integrals, n is the index of the last retained term in (5). Reference values: $e^{0.5} = 1.6487212707$, $\sin 0.3 = 0.2955202067$, $\ln 1.2 = 0.1823215568$, $\sqrt{1.1} = 1.0488088482$, $\int_0^1 e^{-x^2}\,dx = 0.7468241328$, $\mathrm{Si}(1) = 0.9460830704$.

The dependence of convergence speed on the position of the evaluation point is shown in Figure 1. On a logarithmic scale the error curves for $x = 0.2$, 0.5 and 0.8 are close to straight lines whose slopes are proportional to $\ln(|x|/R)$, so the error decreases approximately as $(|x|/R)^N/N$. To reach an accuracy of 10^{-6} the series requires 7 terms at $x = 0.2$, 15 at $x = 0.5$, 42 at $x = 0.8$, 83 at $x = 0.9$ and 660 at $x = 0.99$. At the endpoint $x = 1$ the decay becomes algebraic, with error approximately $1/(2N)$, and after sixty terms the error is still 8.2×10^{-3}. The curves for $x = 0.2$ and $x = 0.5$ level off near 10^{-17}–10^{-16}, which is the rounding floor of double-precision arithmetic: beyond this point additional terms no longer change the computed sum.

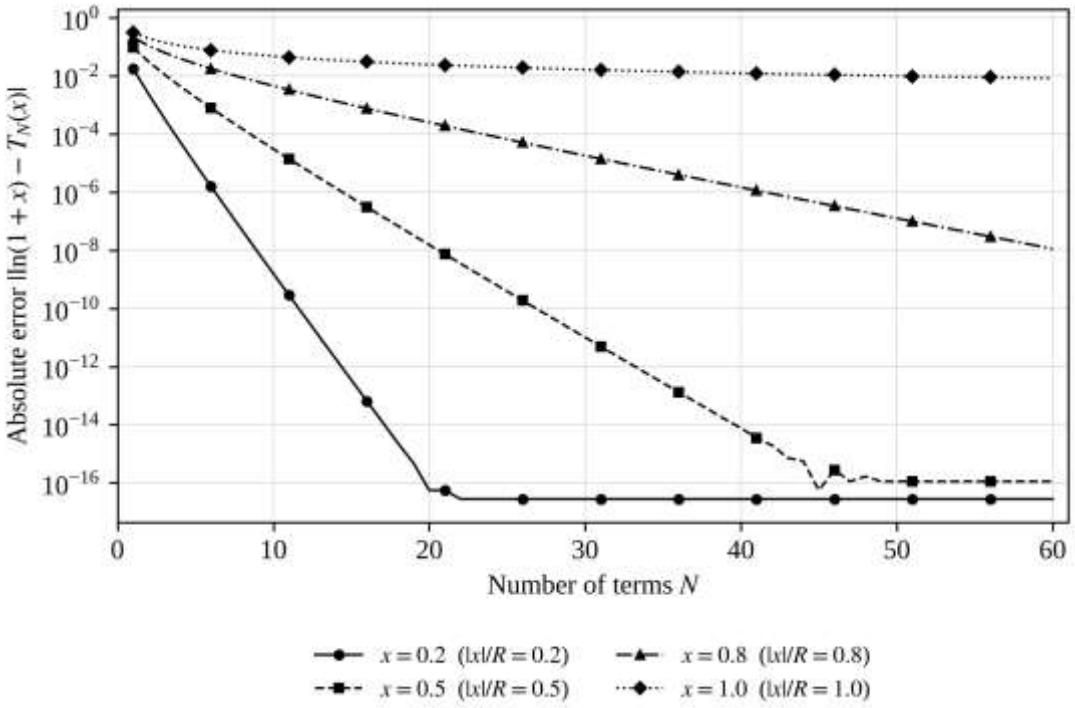

Figure 1. Absolute error of the Mercator series for $\ln(1 + x)$ as a function of the number of terms N at four points with different relative distance $|x|/R$ from the centre.

Figure 2 illustrates the same phenomenon in the variable x for a fixed number of terms. Inside the interval $(-1, 1]$ the Taylor polynomials T_5, T_{10} and T_{25} become indistinguishable from $\ln(1 + x)$ over an expanding central region, the agreement deteriorating only near $x = -1$, where the function itself has a logarithmic singularity. Immediately to the right of $x = 1$ the behaviour reverses: higher-degree polynomials depart from the function sooner and more violently, the deviation from $\ln(1 + x)$ exceeds 0.5 at $x \approx 1.37$ for T_5, at $x \approx 1.25$ for T_{10} and already at $x \approx 1.14$ for T_{25}, with T_{10} falling toward $-\infty$ and T_{25} rising toward $+\infty$. Raising the degree outside the domain of convergence makes the approximation worse, not better.

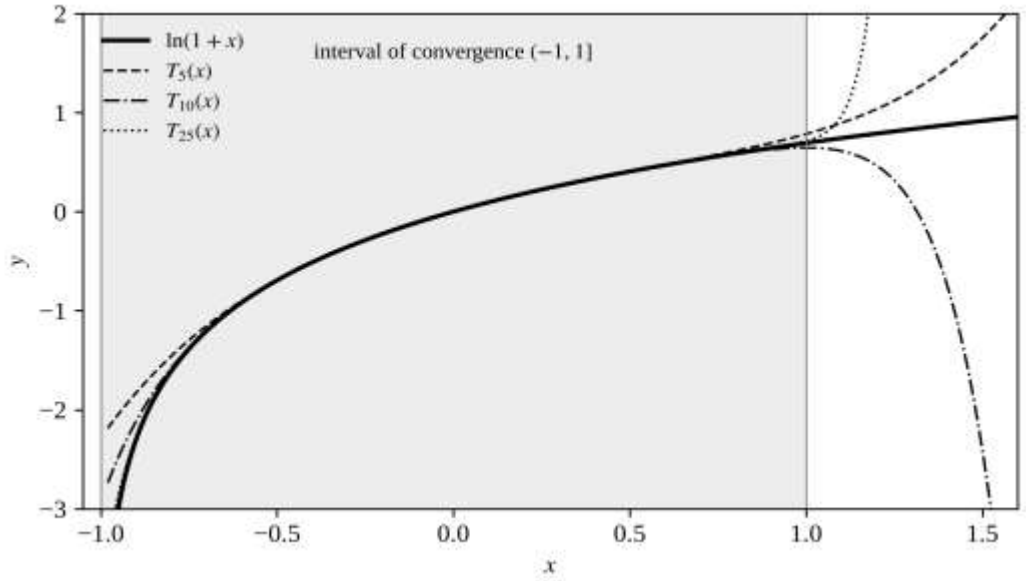

Figure 2. The function $\ln(1 + x)$ and its Taylor polynomials T_5, T_{10} and T_{25}; the shaded band is the interval of convergence $(-1, 1]$.

Practical libraries for elementary functions exploit precisely this behaviour by reducing the argument to a short interval near the expansion point before applying a polynomial approximation [10]. Table 3 quantifies the gain from such reformulation for three constants. Evaluating ln2 from the Mercator series at its endpoint $x = 1$ requires roughly 5×10^5 terms for six correct decimal places and 5×10^{11} terms for twelve, whereas the equivalent series (6), evaluated at the interior point $t = 1/3$ with effective ratio $|t|^2 = 1/9$, needs 6 and 12 terms. Machin's formula reduces the computation of π from about 10^6 and 10^{12} terms of the Leibniz series to 4 and 9 terms. The reference series for e, whose radius is infinite, needs 10 and 15 terms, and the growth from the first to the second tolerance is modest for every series evaluated strictly inside its interval of convergence. More elaborate reformulations of this kind underlie modern multiple-precision algorithms for elementary functions [11].

Table 3. Number of terms required to compute classical constants with absolute error below 10^{-6} and 10^{-12}

Constant	Series used	Evaluation point / effective ratio	Terms for 10^{-6}	Terms for 10^{-12}
$\ln 2$	$\ln(1+x)$, Mercator	$x = 1$ / 1 (endpoint)	$\approx 5 \times 10^5$	$\approx 5 \times 10^{11}$
$\ln 2$	2 artanh t, eq. (6)	$t = 1/3$ / 1/9	6	12
π	4arctan1, Leibniz	$x = 1$ / 1 (endpoint)	$\approx 1 \times 10^6$	$\approx 1 \times 10^{12}$
π	Machin's formula	1/5 and 1/239 / 1/25	4	9
e	e^x at $x = 1$	$x = 1$ / 0 ($R = \infty$)	10	15

Machin-type identities form an entire family of such formulas, in which π is expressed through arctangents of small reciprocals of integers; their efficiency is governed by the size of those arguments relative to the unit radius of convergence of the arctangent series [12]. The data in Table 3 confirm the general rule that emerges from Figure 1: the cost of a Taylor computation is determined far less by the function being approximated than by the ratio $|x - x_0|/R$ at which its series is evaluated.

Discussion

The results support a simple quantitative model of Taylor-based approximation. When the evaluation point lies strictly inside the interval of convergence, the terms of the series are eventually dominated by a geometric sequence with ratio $\rho = |x - x_0|/R < 1$, and the number of terms needed for tolerance ε is approximately

$$N(\varepsilon) \approx \frac{\ln(1/\varepsilon)}{\ln(1/\rho)}. \qquad (7)$$

Formula (7) gives 8.6, 19.9 and 61.9 terms for $\varepsilon = 10^{-6}$ at $\rho = 0.2$, 0.5 and 0.8; the observed values in Figure 1 (7, 15 and 42) are somewhat lower because of the additional factor $1/N$ in the Mercator terms, but the trend is reproduced exactly, including the divergence of $N(\varepsilon)$ as $\rho \to 1$. Two practical conclusions follow. First, reducing ρ divides the work by the factor $\ln(1/\rho_{\text{new}})/\ln(1/\rho_{\text{old}})$, which is large when ρ is close to 1 (moving from $\rho = 0.8$ to $\rho = 0.4$ cuts the work about fourfold), whereas tightening ε by a factor of ten adds only a fixed number $\ln 10/\ln(1/\rho)$ of terms; reducing the distance to the centre is therefore the more powerful lever. Second, at the boundary ($\rho = 1$) the geometric model breaks down and the convergence becomes algebraic, which is why the endpoint series for ln2 and π in Table 3 are of theoretical interest only, even though Abel's theorem guarantees that they converge to the correct values.

The comparison of finite-index radius estimates in Table 1 has a methodological implication. Although the Cauchy–Hadamard formula (1) is the more general criterion, valid when the ratio limit does not exist, its finite-n values converge slowly and can overestimate R by several per cent even at $n = 100$. When numerical evidence is used to estimate the radius of a series whose coefficients are known only numerically,

the ratio estimate should be preferred when the coefficients are regular, and in either case the estimate should be treated as approximate. The direction of approach also varies, from below for $\sum n^2 x^n/2^n$ and from above for the others, so a finite-index estimate cannot be relied upon to give a safe lower bound for R.

The close agreement between true errors and a priori bounds in Table 2 means that, for points well inside the interval of convergence, the remainder estimates can be used directly to choose the number of terms without wasteful overcalculation. The worst overestimate, a factor of about 1.6 for $e^{0.5}$, corresponds to less than one additional term. The situation changes near the boundary, where Lagrange-type bounds become pessimistic and the alternating-series bound (4) becomes the only practical estimate. Figure 1 also shows that rounding error, rather than truncation error, ultimately limits attainable accuracy: once the remainder falls below the unit roundoff, further terms are ineffective. Any choice of N beyond the point at which $|u_{N+1}|$ falls below $\varepsilon_{\text{mach}}|S|$ is wasted effort in double precision.

When the evaluation point cannot be moved closer to the centre, there are alternatives to increasing the number of Taylor terms. Rational approximations built from the same Taylor coefficients, the Padé approximants, frequently converge beyond the disc of convergence of the original series because their poles can mimic the singularities that limit R [13]. Sequence-transformation and extrapolation methods, such as Richardson extrapolation, the Shanks transformation and Wynn's epsilon algorithm, can accelerate slowly convergent series, including the endpoint series considered here, by several orders of magnitude [14]. Finally, when the goal is uniform accuracy over an interval rather than high accuracy at a single point, Chebyshev interpolants are generally superior to Taylor polynomials, since Taylor polynomials concentrate their accuracy near x_0 and lose it toward the ends of the interval, as Figure 2 clearly shows [15]. These methods do not invalidate the Taylor approach; rather, they confirm that the location of singularities, and hence the domain of convergence, is the primary factor governing the quality of any polynomial-type approximation.

On the basis of these observations, a five-step procedure is proposed for teaching and for routine computation. (1) Determine the radius of convergence by formula (2) or (1) and, if necessary, the behaviour at the endpoints. (2) Verify that the evaluation point lies inside the interval and compute the ratio $\rho = |x - x_0|/R$. (3) If ρ is close to 1, reformulate the problem by argument reduction, a change of expansion point, or an algebraic identity such as (6) or Machin's formula, so as to make ρ small. (4) Choose the number of terms using the appropriate remainder bound, with (7) as a quick preliminary estimate. (5) Sum the series in order of decreasing term magnitude, or by Horner's scheme, and compare the result with the bound. The procedure makes explicit the link that the traditional course structure leaves implicit, and every example in this paper can be used as a classroom illustration of it.

The study has limitations. The analysis was restricted to real power series of elementary functions with known reference values and to double-precision arithmetic; series whose coefficients must themselves be computed numerically, and multiple-precision settings in which rounding error is negligible, may display different trade-offs. The experiments also did not address functions of several variables, for which the domain of convergence of a multiple power series is no longer a simple interval or disc.

Conclusion

The domain of convergence of a power series is not only a theoretical property but the decisive factor in the efficiency of Taylor-based approximate computation. Inside the interval of convergence, the error of a Taylor approximation decreases approximately geometrically with ratio $|x - x_0|/R$, the a priori remainder bounds exceed the true errors by no more than a factor of about 1.6, and a few terms are sufficient for six to twelve correct digits, as shown for $e^{0.5}$, $\sin 0.3$, $\ln 1.2$, $\sqrt{1.1}$ and two non-elementary integrals. As the evaluation point approaches the boundary, the required number of terms grows without bound; at the

boundary convergence becomes algebraic, and beyond it the partial sums diverge, with higher-degree polynomials performing worse. Reformulating a problem so that its series is evaluated close to the centre of the interval of convergence reduces the computational cost by up to eleven orders of magnitude, as illustrated by the computation of $\ln 2$ and π. Integrating the analysis of the domain of convergence into the procedure for choosing the number of terms, as in the five-step scheme proposed here, therefore improves both the teaching of the subject and the reliability of practical computation. Further work may extend the analysis to Padé and Chebyshev approximations and to functions of several variables.

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